MOUNT LAWLEY SENIOR HIGH SCHOOL

Semester 2 Examination, 2011

Question/Answer Booklet

MATHEMATICS SPECIALIST
MAS 3C/3D

Section One
Calculator-free

NAME Key

Time allowed for this section

Reading time before commencing work: 5 minutes
Working time for paper: 50 minutes

Material required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet

To be provided by the candidate

Standard items: pens, pencils, pencil sharpener, eraser, correction fluid, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Section One: Calculator-free **(40 Marks)**

This section has **six (6)** questions.
Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(7 marks)**

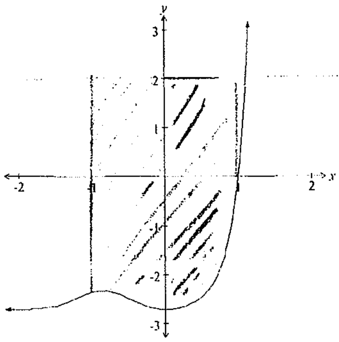
Consider the function $f(x)=x^2e^{x^3}-e$.

a) Determine $\int f(x)\,dx$. [1]

$$\frac{e^{x^3}}{3} - ex + C$$

no penalty!

The graph of $y=f(x)$ is shown.



b) Carefully shade the area contained between $y=f(x)$, the lines $x=-1$, $x=1$, and $y=2$. [1]

c) Write down an integral whose value represents the shaded area. [2]

$$\int_{-1}^{1} 2 - x^2e^{x^3} + e \; dx$$

d) Prove the shaded area is exactly equal to $4+\dfrac{1+5e^2}{3e}$. [3]

$$= \left[2x + ex - \frac{e^{x^3}}{3}\right]_{-1}^{1}$$

$$= \left(2 + e - \frac{e}{3}\right) - \left(2(-1) + e(-1) - \frac{e^{-1}}{3}\right)$$

$$= 4 + 2e - \frac{e}{3} + \frac{1}{3e}$$

$$= 4 + \frac{5e}{3} + \frac{1}{3e}$$

$$= 4 + \frac{5e^2+1}{3e}$$

Question 2 **(5 marks)**

(a) Find the equation of the tangent to the curve $x^3 - y^3 = 2$ at the point on the curve where $x = 1$. [2]

diff. w.r.t x $\Rightarrow$ $3x^2 - 3y^2 y' = 0$

ie $\frac{dy}{dx} = \frac{x^2}{y^2}$

when $x = 1$, $1^3 - y^3 = 2 \Rightarrow y = -1$

$\therefore \frac{dy}{dx} = 1$

Eqⁿ is $y = x + c$

at $(1,-1) \Rightarrow c = -2$

Eqⁿ $y = x - 2$ ✓ ✓

(b) Evaluate $\int_1^{e^2} \frac{(\ln x)^2}{x}\, dx$. [3]

$$= \int_1^{e^2} \underbrace{(\ln x)^2}_{f(x)} \cdot \underbrace{\frac{1}{x}}_{f'(x)}\, dx$$

$$= \left.\frac{(\ln x)^3}{3}\right]_1^{e^2}$$ ✓

$$= \frac{(\ln e^2)^3}{3} - \frac{(\ln 1)^3}{3}$$

$$= \frac{2^3}{3} - 0$$

$$= \frac{8}{3}$$ ✓

OK

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x}dx$

$x = 1$, $u = 0$ ✓

$x = e^2$, $u = 2$ ✓

$$\therefore \int_1^{e^2} (\ln x)^2 \cdot \frac{1}{x}\, dx$$

$$= \int_0^2 u^2\, du$$

$$= \left.\frac{u^3}{3}\right]_0^2$$

$$= \frac{8}{3}$$ ✓

Question 3 **(6 marks)**

The transformation matrix $M = \begin{bmatrix} -2 & 1 \\ a & b \end{bmatrix}$.

M represents a shear of factor k parallel to the y-axis followed by a rotation of 90° clockwise.

(a) Use properties of the two transformations to explain why $|M| = 1$ (ie Det M = 1). [1]

Area of image = |Det M| × Area object

Shear : area is preserved } area of image equals area of object
rotation : area is preserved }

$\therefore |M| = 1$.

(b) Determine the values of a, b and k. [3]

Shear // to Y axis $\Rightarrow \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

rotation 90° clockwise $\Rightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$\therefore \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ a & b \end{bmatrix} \quad ✓$$

$$\text{ie} \quad \begin{bmatrix} k & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ a & b \end{bmatrix}$$

$$\therefore k = -2, \quad a = -1, \quad b = 0 \quad ✓✓$$

The point P is transformed by M to the point $(8,3)$.

(c) Determine the coordinates of P. [2]

$$P(x,y) \xrightarrow{M} (8,3) \qquad M^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 8 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \quad ✓$$

$\therefore$ P is $(-3,2)$ ✓

Question 4 **(8 marks)**

Consider the identity $2i\ sin(n\theta) = z^n - \frac{1}{z^n}$ where $z = \text{cis}\theta$.

(a) By initially letting $n = 1$, show how to use the identity to prove that:

$$sin^3\theta = \frac{3sin\theta - sin(3\theta)}{4}.$$ [5]

$n = 1 \Rightarrow \quad 2i \sin\theta = z - \frac{1}{z}$ ✓

$\therefore (2i \sin\theta)^3 = \left(z - \frac{1}{z}\right)^3$ ✓

ie $8i^3 \sin^3\theta = z^3 - 3.z^2\frac{1}{z} + 3z.\frac{1}{z^2} - \frac{1}{z^3}$ ✓

ie $-8i \sin^3\theta = \left(z^3 - \frac{1}{z^3}\right) - 3\left(z + \frac{1}{z}\right)$ ✓

ie $-8i \sin^3\theta = 2i \sin(3\theta) - 3\ 2i \sin\theta$

$\therefore \sin^3\theta = \frac{2\cancel{i}}{-8\cancel{i}}\sin(3\theta) - \frac{6\cancel{i}}{-8\cancel{i}}\sin\theta$ ✓

$= -\frac{\sin(3\theta)}{4} + \frac{3\sin\theta}{4}$

$\therefore \sin^3\theta = \frac{3\sin\theta}{4} - \frac{\sin(3\theta)}{4}$

ie $\sin^3\theta = \frac{3\sin\theta - \sin(3\theta)}{4}$

Question 4 (continued)

Hence,

(b) evaluate $\int_0^{\pi} 9\,sin\,x - 12\,sin^3 x\,dx$. [3]

Now $4\sin^3 x = 3\sin x - \sin 3x$

$\therefore \sin 3x = 3\sin x - 4\sin^3 x$ ✓

ie $3\sin 3x = 9\sin x - 12\sin^3 x$

$\therefore \int_0^{\pi} 9\sin x - 12\sin^3 x \, dx$

$= \int_0^{\pi} 3\sin 3x \, dx.$

$= \cancel{3}. \left[-\dfrac{\cos 3x}{\cancel{3}} \right]_0^{\pi}$ ✓

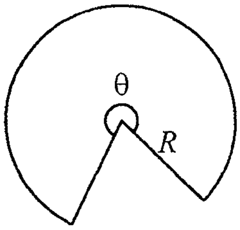
$= -\cos 3\pi - (-\cos 0)$

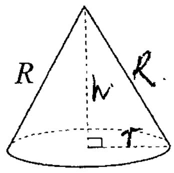
$= -(-1) - (-1)$

$= \underline{2}$ ✓

Question 5 **(7 marks)**

A minor sector of angle $2\pi - \theta$ is removed from a circular piece of paper of radius R. The two straight edges of the remaining major sector are pulled together to form a right circular cone, with a slant height of R.





(a) Show that the volume of the cone is given by $V = \dfrac{R^3\theta^2\sqrt{4\pi^2 - \theta^2}}{24\pi^2}$. [3]

V of Cone $= \frac{1}{3}\pi r^2 h$

Arc length of paper $= R\theta$

Arc length of Paper = circumference of base of cone.

$\therefore R\theta = 2\pi r \Rightarrow r = \frac{R\theta}{2\pi}$ ✓

Also $R^2 = h^2 + r^2$

$\therefore R^2 = h^2 + \left(\frac{R\theta}{2\pi}\right)^2$

ie $h^2 = R^2 - \frac{R^2\theta^2}{4\pi^2}$

ie $h^2 = R^2\left(1 - \frac{\theta^2}{4\pi^2}\right)$.

ie $h^2 = R^2\left(\frac{4\pi^2 - \theta^2}{4\pi^2}\right)$

$\therefore h = R\sqrt{\frac{4\pi^2 - \theta^2}{4\pi^2}}$

ie $h = R\,\frac{\sqrt{4\pi^2 - \theta^2}}{2\pi}$ ✓✓

$V = \frac{1}{3}\pi \cdot \left(\frac{R\theta}{2\pi}\right)^2 \times R\,\frac{\sqrt{4\pi^2 - \theta^2}}{2\pi}$

ie $V = \frac{R^3\theta^2\sqrt{4\pi^2 - \theta^2}}{24\pi^2}$

Question 5 (continued)

Assuming the radius, R, of the circular piece of paper to be fixed,

(b) find the exact value of θ which maximises the volume of cone. [4]

$$V = \frac{R^3}{24\pi^2} \cdot \theta^2 \sqrt{4\pi^2 - \theta^2}$$

$$\frac{dV}{d\theta} = \frac{R^3}{24\pi^2}\left[2\theta\sqrt{4\pi^2-\theta^2} + \theta^2 \cdot \frac{1}{2\sqrt{4\pi^2-\theta^2}} \cdot (-2\theta)\right] \quad ✓✓$$

$$\frac{dV}{d\theta} = \frac{R^3}{24\pi^2}\left[2\theta\sqrt{4\pi^2-\theta^2} - \frac{\theta^3}{\sqrt{4\pi^2-\theta^2}}\right]$$

$$\frac{dV}{d\theta} = 0 \Rightarrow 2\theta\sqrt{4\pi^2-\theta^2} = \frac{\theta^3}{\sqrt{4\pi^2-\theta^2}}. \quad ✓$$

$$\therefore 2(4\pi^2-\theta^2) = \theta^2$$

$$\text{ie } 8\pi^2 - 2\theta^2 = \theta^2.$$

$$\therefore 3\theta^2 = 8\pi^2$$

$$\theta^2 = \frac{8\pi^2}{3}$$

$$\therefore \theta = \sqrt{\frac{8\pi^2}{3}} \quad ✓$$

$$= \frac{\sqrt{8}\,\pi}{\sqrt{3}}$$

$$= \frac{2\sqrt{2}\,\pi}{\sqrt{3}}.$$

Question 6 **(8 marks)**

The Argand diagram below shows the complex number $z_1 = a + ib$ as a position vect r with a and b having integer values.

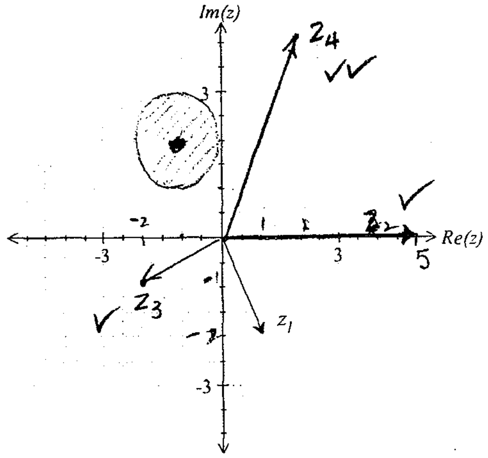
$z_1 = 1 - 2i$

(a) On the same diagram plot and label the complex numbers given by:

$z_2 = z_1 \times \bar{z}_1 = (1-2i)(1+2i)$
$= 5$

$z_3 = i^3 \times z_1 = -i(1-2i)$
$= -i - 2$

$z_4 = 10(z_1)^{-1}$
$= 10 \cdot \frac{1}{1-2i} \cdot \frac{1+2i}{1+2i}$
$= \frac{10(1+2i)}{5}$
$= 2 + 4i$



[4]

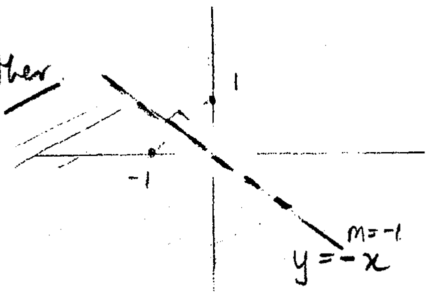
(b) On the same diagram sketch the region given by $|z + z_1| \leq \cancel{2}\ 1$. [2]

$|z + 1 - 2i| \leq \cancel{2}\ 1$

Shaded circle ✓
position centre & Rad. ✓

(c) Determine the Cartesian equation described by $|z + 1| < |z - i|$. [2]

either



$y < -x$
✓✓

or. $|z+1| < |z-i|$

put $z = x + iy$

$|(x+1) + iy| < |x + i(y-1)|$

$\therefore (x+1)^2 + y^2 < x^2 + (y-1)^2$

$\cancel{x^2} + 2x + \cancel{1} + \cancel{y^2} < \cancel{x^2} + \cancel{y^2} - 2y + \cancel{1}$

$2x < -2y$

$\therefore x < -y$ or $y < -x$

✓✓

End of Section One